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Fisher information and bounds to the entropy increase

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With reference to systems obeying conservation of flow, for which a probability density function may be used in their description, it is shown that the interesting relationship established by Nikolov and Frieden (NF) between entropy increase and Fisher information [Phys. Rev. E **49**, 4815 (1994)] holds only in special cases. We will here (i) exhibit a counterexample (to the hypothetical NF relationship) and (ii) derive a valid, Fisher's information-related upper bound to entropy increases.

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I. INTRODUCTION

Consider a system described by a probability distribution $p(\mathbf{r}|t)$. Particle position $\mathbf{r}=(x,y,z)$ is random, and specified by a conditional probability law $p(\mathbf{r}|t)$, representing the probability of a particle at position $\mathbf{r}$, conditional upon (at) time t. The broader this is, the more equally probable are all $\mathbf{r}$ values, which entails a higher degree of disorder for the system at that time [1].

Normalization holds at each time t

$$\int p(\mathbf{r}|t)d\mathbf{r}=1 , \tag{1}$$

and, following Nikolov and Frieden (NF) [1], we assume that the temporal evolution of the probability distribution is governed by a continuity equation

$$\frac{\partial p}{\partial t}+\nabla\cdot\mathbf{P}=0 , \tag{2}$$

where $\mathbf{P}(\mathbf{r},t)$ is the probability density current, whose exact nature depends upon the application [1]. Only systems for which the boundary conditions discussed by NF are here to be studied. The *NF*-boundary conditions read

$$p(\mathbf{r}|t)|_{\text{boundaries}}=0 , \tag{3}$$

$$\mathbf{P}(\mathbf{r},t)|_{\text{boundaries}}=0 , \tag{4}$$

$$\mathbf{P}\ln p|_{\text{boundaries}}=0 , \tag{5}$$

and it is also assumed that, if the boundaries are at infinity, the quantities $p,\mathbf{P}$, and $\mathbf{P}\ln p$ tend to zero in a faster fashion than $1/r^2$ [1]. We reiterate that, in what follows, we shall concern ourselves *only* with systems for which the NF-boundary conditions apply.

The Fisher information I is defined as [2,3]

$$I(t)=\int d\mathbf{r}\frac{\nabla p\cdot\nabla p}{p} , \tag{6}$$

while the Boltzmann-Shannon entropy is, of course,

$$H(t)=-\int d\mathbf{r}\, p(\mathbf{r}|t)\ln p(\mathbf{r}|t) . \tag{7}$$

In Ref. [1] NF expounded the notable idea that the rate of increase of Shannon's entropy is bounded and that the concomitant bound is related to Fisher's information. This bound is given by the following inequality [1]:

$$\frac{dH}{dt}\le\tfrac{1}{6}I(t)\frac{d}{dt}\langle r^2\rangle , \tag{8}$$

where $\langle r^2\rangle=\int d\mathbf{r}\, p(\mathbf{r}|t)r^2$.

A corollary to (8) is that, whenever $dH/dt\ge 0$,

$$\frac{d}{dt}\langle r^2\rangle\ge 0 . \tag{9}$$

The remarkable NF insight of relating dH/dt to $I(t)$ needs a little additional work, however. We will here show that Eqs. (8) and (9) do not always hold. A counterexample will first be exhibited and then we shall derive an inequality relating dH/dt to $I(t)$. Contrary to what happens with (8), the result of our inequality is always valid.

II. A COUNTEREXAMPLE

Consider a system for which

$$\mathbf{P}=\mathbf{v}p , \tag{10}$$

so that the continuity equation adopts the appearance

$$\frac{\partial p}{\partial t}+\nabla\cdot(\mathbf{v}p)=0 . \tag{11}$$

We assume that our system is of such a nature that the velocity field is given by

$$v_i=\alpha_i r_i;\quad \alpha_i \in \mathbb{R};\quad i=1,2,3\ . \tag{12}$$

(The reader interested in the motivation for this example is referred to the Appendix.)

It is now easy to verify that the continuity equation (11) admits, in the present circumstances, solutions of the Gaussian form

$$p(\mathbf{r}|t)=\frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3}\exp\left[-\frac{1}{2}\sum_{i=1}^{3}\frac{r_i^2}{\sigma_i^2}\right], \tag{13}$$

with a time dependent dispersion $\sigma(t)$ such that

$$\frac{d\sigma_i}{dt}=\alpha_i\sigma_i(t),\quad i=1,2,3\ . \tag{14}$$

It is clear that, for this system, NF-boundary conditions are fulfilled. In the present situation our main quantities, Shannon's H and Fisher's I, acquire the simple forms

$$H=\tfrac{3}{2}[1+\ln(2\pi)]+\sum_{i=1}^{3}\ln\sigma_i\ , \tag{15}$$

and

$$I=\sum_{i=1}^{3}\frac{1}{\sigma_i^2}\ . \tag{16}$$

The main ingredients entering Eqs. (8) and (9) read here

$$\frac{dH}{dt}=\alpha_1+\alpha_2+\alpha_3\ , \tag{17}$$

where Eqs. (14) and (15) have been used. Further, (14) leads to

$$\frac{d}{dt}\langle r^2\rangle=2\sum_{i=1}^{3}\alpha_i\sigma_i^2\ . \tag{18}$$

The right-hand side (rhs) of Eq. (8) is then of the form

$$\tfrac{1}{6}I(t)\frac{d}{dt}\langle r^2\rangle=\frac{1}{3}\left[\sum_{i=1}^{3}\frac{1}{\sigma_i^2}\right]\left[\sum_{i=1}^{3}\alpha_i\sigma_i^2\right], \tag{19}$$

and we proceed now to show that in the following two instances we can detect violations to either Eq. (8) or Eq. (9).

(a) We take $\alpha_1=1,\alpha_2=\alpha_3=0$. From Eqs. (16)–(19) one immediately ascertains that

$$\frac{dH}{dt}=1\ , \tag{20}$$

and

$$\tfrac{1}{6}I(t)\frac{d}{dt}\langle r^2\rangle=\frac{1}{3}\left[1+\frac{\sigma_1^2}{\sigma_2^2}+\frac{\sigma_1^2}{\sigma_3^2}\right]. \tag{21}$$

Now, in that *region* of the $(\sigma_1,\sigma_2,\sigma_3)$ space for which

$$\begin{aligned}&\sigma_1<\sigma_2\ ,\\&\sigma_1<\sigma_3\ ,\end{aligned} \tag{22}$$

it is clear that

$$\tfrac{1}{6}I(t)\frac{d}{dt}\langle r^2\rangle<\frac{dH}{dt}\ , \tag{23}$$

and Eq. (8) *does not hold* within that region.

(b) We assume that $\alpha_1=\alpha_2=1$, and $\alpha_3=-1$. It is seen that Eq. (20) holds here as well, while from Eq. (18) we obtain

$$\frac{d}{dt}\langle r^2\rangle=2(\sigma_1^2+\sigma_2^2-\sigma_3^2)\ , \tag{24}$$

and it is obvious that, in the *region* of the $(\sigma_1,\sigma_2,\sigma_3)$ space for which

$$\sigma_1^2+\sigma_2^2<\sigma_3^2\ , \tag{25}$$

$(d/dt)\langle r^2\rangle$ is *a negative quantity,* notwithstanding the fact that $dH/dt>0$. Thus, Eqs. (8) and (9) are *both violated.*

Admittedly, this is a very simple (and even artificial) example. However, its mere existence invalidates (8) and (9) as *general* assertions.

III. A CORRECT BOUND TO THE ENTROPY INCREASE

From the continuity equation (2) and the definition (7) of H one easily finds [cf. Eq. (21) of [1]]

$$\frac{dH}{dt}=-\int\frac{\mathbf{P}}{\sqrt{p}}\cdot\frac{\nabla p}{\sqrt{p}}d\mathbf{r}\ , \tag{26}$$

which will be the starting point of the present considerations. We first remind the reader that if one deals with two vector fields $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ the following inequalities ensue:

$$\begin{aligned}\left|\int d\mathbf{r}\ \mathbf{A}\cdot\mathbf{B}\right|&\le\int d\mathbf{r}|\mathbf{A}\cdot\mathbf{B}|\\&\le\int d\mathbf{r}|\mathbf{A}|\,|\mathbf{B}|\\&\le\left[\int d\mathbf{r}|\mathbf{A}|^2\right]^{1/2}\left[\int d\mathbf{r}|\mathbf{B}|^2\right]^{1/2},\end{aligned} \tag{27}$$

where the last inequality is that of Schwartz's. Setting now $\mathbf{A}=\mathbf{P}/\sqrt{p}$ and $\mathbf{B}=\nabla p/\sqrt{p}$, we are immediately led to

$$\left|\frac{dH}{dt}\right|\le\left|\int d\mathbf{r}\frac{P^2}{p}\right|^{1/2}\left|\int d\mathbf{r}\frac{\nabla p\cdot\nabla p}{p}\right|^{1/2}, \tag{28}$$

which, with the definition (a special case of which was already used in the counterexample)

$$\mathbf{v}=\frac{\mathbf{P}}{p}\ , \tag{29}$$

and the expression (6) for Fisher's information, leads to

$$\left|\frac{dH}{dt}\right|\le\langle v^2\rangle^{1/2}I^{1/2}\ , \tag{30}$$

where

$$\langle v^2\rangle=\int d\mathbf{r}\,pv^2\ , \tag{31}$$

is, of course, the expectation value of the squared "velocity field" v^2.

Equation (30) constitutes the desired bound to entropy increases, expressed in terms of Fisher's information, which should be used instead of Eq. (8) (that holds only in special situations).

IV. DISCUSSION

What is the problem with Eq. (8)? In [1], NF's attempt to derive an upper bound to entropy increases by evaluating the rhs of Eq. (26) under the assumption that $\mathbf{P}=a\nabla p$ [*their* Eq. (23)], with a a function of t (independent of $\mathbf{r}$). An equation which looks like (8) (ours) is, in such a fashion, arrived at. But the quantities I and $(d/dt)\langle r^2\rangle$ appearing therein do not correspond to the actual probability distribution p. They correspond, rather, to a *fictitious* distribution that satisfies the assumption referred to above. Thus, the two sides of Eq. (8) end up being expressed in terms of quantities evaluated with *different* probability distributions.

However, the main point of NF is a valid (and quite important) one: *a bound to the entropy increase can be given in terms of Fisher's information.* Our equation (30) constitutes an expression for such a bound and complements thus the NF's notable finding of Ref. [1].

APPENDIX

Any autonomous system of N ordinary differential equations

$$\frac{d\mathbf{x}}{dt}=\mathbf{v}(\mathbf{x}),\quad \mathbf{x},\mathbf{v}\in\mathbb{R}^n\ , \tag{A1}$$

can be related to a Liouville equation of the form [4]

$$\frac{\partial p}{\partial t}+\nabla\cdot(\mathbf{v}p)=0\ , \tag{A2}$$

where $p(\mathbf{x},t)$ is, at time t, the density of points in phase space [4]. This Liouville equation describes the temporal evolution of an *ensemble* of systems, distributed in phase space with density p, each of which evolves as prescribed by (A1). Obviously, (A2) is a continuity equation of the form

$$\frac{\partial p}{\partial t}+\nabla\cdot\mathbf{P}=0\ , \tag{A3}$$

where

$$\mathbf{P}=\mathbf{v}p\ . \tag{A4}$$

Here the velocity field $\mathbf{v}=\mathbf{P}/p$ depends exclusively upon the position (in phase space) $\mathbf{x}$ and *not* upon the time, as in our counterexample. Such a situation can be encountered, for instance, in [5].

The form (A2) was the one originally employed by Liouville [4] and is valid for *any* autonomous system of ordinary differential equations, while most textbooks deal with a less general Liouville equation, restricted to Hamiltonian systems.

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